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On Schrödinger semigroups and related topics

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Abstract

This paper deals with two related subjects. In the first part, we give generation theorems, relying on (weak) compactness arguments, for perturbed positive semigroups in general ordered Banach spaces with additive norm on the positive cone. The second part provides new functional analytic developments on semigroup theory for Schrödinger operators in L^p spaces with (L^1) Δ-bounded potentials without restriction on the (L^1) Δ-bound. In particular, our formalism enlarges a priori the classical Kato class and its subsequent refinements. The connection with form-perturbation theory is also dealt with.

Keywords: Schrödinger operator; Kato class; Positive semigroups; Weak compactness; Essential self-adjointness; Form-perturbation

1. Introduction

This paper provides some results on semigroup theory and on Schrödinger operators. The first part deals with new generation theorems (of perturbative type) of positive semigroups in general ordered Banach spaces with additive norm on the positive cone. We note that such Banach spaces cover $L^1(\mu)$ spaces (or spaces of bounded measures) and also some other ordered Banach spaces of practical interest but without lattice structure such as the Banach space of trace class operators in a Hilbert space where our results can apply to quantum dynamical semigroups. Our generation theorems rely on (weak) compactness arguments.

The second part of the paper provides new functional analytic developments on Schrödinger operators $-\Delta - V$ in $L^p(R^N)$ where Δ is the Laplacian and V is an unbounded multiplica-

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tion operator by a positive measurable function V which does not necessarily fall within known classes of potentials (we use the same symbol for the function V and the multiplication operator by V). In the literature, this picture refers to Schrödinger operators with *negative* potentials, i.e. $-V \leqslant 0$ is the potential; we have also dropped the classical coefficient $\frac{1}{2}$ in front of the Laplacian. We note that the case of positive (or equivalently bounded below) potentials is well understood under very weak assumptions (e.g. $V \in L^1_{\mathrm{loc}}(R^N - F)$ where F is a closed set with zero-Lebesgue measure) and a general m-accretivity theory is available. On the other hand, it is well known that the treatment of negative potentials requires some "smallness" condition; typically the relative bound of V with respect to Δ (the Δ-bound) must be small enough or, at least, the potential must be form-small in L^2 with respect to $-\Delta$. The state of the art by the end of the sixties is comprehensively covered by Schechter's monograph [35]. The subsequent literature, influenced to some extent by some seminal papers such as [16,18,36,38], is really considerable: For the next two decades (and without any pretense to completeness), we refer to the classical books [33, Chapter X], [20, Chapter V] and to the papers [1,9,14,17,19,21–25,30,37,39,40,43] and references therein; some works rely also on probabilistic tools, e.g. [1,9,14,22]. (We mention that complex potentials are also dealt with; see e.g. [8].) An extensive list of references on Schrödinger operators is given in the more recent review paper by B. Simon [41]. We note that our paper deals essentially with one aspect of the subject which is more or less connected to essential self-adjointness. To this end, we provide a systematic semigroup theory for Schrödinger operators in L^p spaces. Since the understanding of positive potentials, i.e. *absorption* semigroups, is essentially complete (see [3,43] and references therein), the present paper focuses mainly on negative potentials; we point out that we could as well add a positive singular (e.g. L^1_{loc}) potential, see Remark 12. We provide a general theory which improves in several respects the literature on the subject; in particular the concept of "*smallness*" of negative potentials with respect to the Laplacian is finely revisited. The general philosophy behind this work is that a great deal of mathematical properties of Schrödinger operators on L^p spaces is in a sense "already contained" in the L^1 theory; this gives the L^1 setting a special status. The role of L^1 appears also (but differently) in the context of spectral theory of Schrödinger operators [12]. Our general strategy is the following: We give first a very general generation theorem for $A_1 := \Delta + V$ in L^1-spaces for Δ-bounded potential in L^1-sense (as a consequence of a perturbation result by W. Desch [13]). This theorem relies on the optimal assumption

$$\delta := \lim_{\lambda \to +\infty} r_\sigma\left[V(\lambda - \Delta)^{-1}\right] < 1 \tag{1}$$

which is much weaker than the usual "Δ-smallness" assumption

$$\lim_{\lambda \to +\infty} \left\| V(\lambda - \Delta)^{-1} \right\|_{\mathcal{L}(L^1)} < 1$$

occurring in the literature. In particular, thanks to the functional analytic results of the first part of the paper, this theorem may be used (i.e. (1) holds) under suitable *weak* compactness assumptions. This allows us to enlarge a priori the known classes of potentials. As far as we know, the idea to get round "Δ-smallness" assumptions by means of weak compactness arguments in L^1 appears here for the first time and improves our understanding of certain aspects of Schrödinger operators even though it is known (see e.g. [1,40]) that most physical potentials have zero Δ-bound. (We point out also that for the potentials with radial symmetry, Δ-boundedness in L^1

sense implies zero Δ-bound of the potential; see [40, Proposition A.2.5] and the comment coming after.)

By a classical symmetry argument, in the spirit of [21], the corresponding semigroup interpolates on L^p spaces and provides us with "Schrödinger semigroups" $\{S_p(t); t \geqslant 0\}$ with generators A_p in L^p spaces. In particular, we capture a very general self-adjoint semi-bounded Schrödinger operator A_2 in L^2. We also provide connections between our formalism and form-perturbation theory in L^2 by showing that (1) implies that V is form-bounded with respect to $-\Delta$ in L^2 with relative form-bound less than or equal to δ. In particular, under (1), we show that A_2 is nothing but $\Delta \dotplus V$ (form-sum). A conjecture on a *characterization* of form-smallness in L^2 of negative potentials V with respect to $-\Delta$ in terms of (1) is also given. The domains of the generators A_p are precisely characterized and practical cores are given. Our formalism is general, self-contained and most of our results are new. We note also that Schrödinger operators with magnetic fields can be dealt with by combining additional domination arguments [29]. Finally, we mention that, without extra mathematical cost, some of our results can be stated in abstract $L^p(\mu)$ spaces for general positive symmetric semigroups (see Remark 25); however, for the simplicity of statements, we have prefered to restrict ourselves to the Laplacian in the whole space. This paper is an expanded version (with applications) of the preprint [28]. It is organized as follows.

In Section 2, we recall a perturbation theorem by W. Desch [13] in ordered Banach spaces X with additive norm on X_+. In Section 3, we prove a result on the spectral radius of positive operators in ordered Banach spaces (without necessarily a lattice structure) and show how Desch's perturbation theorem applies under suitable (weak) compactness assumptions. In Section 4, we show how Desch's perturbation theorem, applied to the Schrödinger operator in $L^1(R^N)$

$$\begin{cases} A_1 : f \in D(\Delta_1) \to \Delta f + Vf \in L^1(R^N), \\ D(\Delta_1) = \{\varphi \in L^1(R^N);\ \Delta\varphi \in L^1(R^N)\}, \end{cases} \tag{2}$$

provides an optimal generation theorem under assumption (1). In particular, we show that if $V = V_1 + V_2$ where the relative bound of V_2 with respect to Δ is less than 1 and V_1 is Δ-weakly compact then A_1 is a generator. We also characterize this weak compactness assumption in terms of the potential V_1. In particular, such an assumption is satisfied if V_1 is "small at infinity" in the sense

$$\sup_{y\in R^N} \int_{|x|\geqslant c} G_1(x-y)V_1(x)\,dx \to 0 \quad \text{as } c \to \infty$$

(e.g. if $y \in R^N \to \int_{|x|\geqslant c} G_1(x-y)V_1(x)\,dx$ is continuous and goes to zero as $|y| \to \infty$) where $G_1(\cdot)$ is the Bessel kernel of $(1-\Delta)^{-1}$ and, for any ball B with finite radius centered at zero,

$$\lim_{|\Omega|\to 0,\, \Omega\subset B} \sup_{x\in B} \int_{\Omega} \left|g_N(x-y)\right| V_1(y)\,dy \to 0,$$

where $g_N(\cdot)$ denotes a fundamental solution of the Laplacian and $|\Omega|$ is the Lebesgue measure of Ω. (We note that the last assumption on V_1 is weaker than the membership to the local class K_N^{loc} defined in [1, p. 210]; see Remark 20 below.) Since *no* condition on the bound of V_1 with respect to Δ is required, this result enlarges a priori the classical Kato class and subsequent refinements; see Remarks 9–11 and 20, Theorems 13, 15, 17 and Proposition 19. By a symmetry argument, the generation result in $L^1(R^N)$ provides generation results in $L^p(R^N)$ $(1 \leqslant p < \infty)$ and the generator A_p (i.e. the Schrödinger operator) of the corresponding Schrödinger semigroup $\{S_p(t); t \geqslant 0\}$ turns out to be the closure of $\Delta + V : \Xi_p \to L^p(R^N)$ where

$$\Xi_p := \{f \in L^p(R^N) \cap D(\Delta_1);\ \Delta f + Vf \in L^p(R^N)\}.$$

In particular, the closure A_2 of $\Delta + V : \Xi_2 \to L^2(R^N)$ is a self-adjoint semi-bounded operator. We also show (in Section 5) that (1) implies that V is form-bounded with respect to $-\Delta$ in L^2 with relative form-bound less than or equal to δ and prove that A_2 is nothing but $\Delta \dotplus V$ (form-sum). We note that, a priori, V is not (L^p) Δ-bounded for $p > 1$. We show that $\mathcal{C}_c^\infty(R^N)$ (the C^∞ functions with compact supports) is a core for A_1; this result is known if the (L^1) relative Δ-bound of V is < 1 [43]. Note that the potential V being (only) locally integrable, $\mathcal{C}_c^\infty(R^N)$ is not a priori contained in the domain of A_p for $p > 1$. The fact that $\mathcal{C}_c^\infty(R^N)$ is a core for A_2 if $V \in L^2_{\mathrm{loc}}(R^N)$ and the (L^1) relative Δ-bound of V is zero is a classical result by T. Kato [18]; this result was generalized later to any $p > 1$ and $V \in L^p_{\mathrm{loc}}(R^N)$ such that the (L^1) relative Δ-bound of V is < 1 [25]. In our general setting with $p > 1$, we show that if $V \in L^p_{\mathrm{loc}}(R^N)$ then $\mathcal{C}_c^\infty(R^N)$ is a core for A_p under the following regularity assumption: For all $g \in \mathcal{S}(R^N)$ (the Schwartz class) the solution f to the problem

$$\lambda f - \Delta f - Vf = g, \quad f \in D(A_1)$$

(for large λ) has a gradient $\nabla f \in L^p(R^N)$. (Note that this regularity assumption above is always true in one dimension.) In particular, this assumption is satisfied for $p = 2$ since we show (in Section 5) by form-perturbation theory that

$$D(A_2) \subset W^{1,2}(R^N).$$

Since $D(A_1) \subset W^{1,1}(R^N)$ then it follows that if $V \in L^2_{\mathrm{loc}}(R^N)$ then $D(A_p) \subset W^{1,p}(R^N)$ for all $1 \leqslant p \leqslant 2$ and consequently $\mathcal{C}_c^\infty(R^N)$ is a core for A_p. However if $p > 2$ or if $1 < p < 2$ with $V \in L^p_{\mathrm{loc}}(R^N) \backslash L^2_{\mathrm{loc}}(R^N)$ then the above regularity hypothesis seems to require further assumptions; thus we show that this hypothesis is satisfied for $1 \leqslant p < \frac{N}{N-2}$ if the potential is "smooth" in the following sense: for each $\varphi \in D(\Delta_1) \cap L^\infty(R^N)$

$$\sup_{|h| \leqslant 1, h \neq 0} \left\| \left(\frac{V_h - V}{|h|}\right)\varphi \right\|_{L^1(R^N)} < \infty,$$

where $V_h : x \to V(x + h)$; the last condition itself is satisfied if for instance the potential V belongs to $W^{1,r}(R^N) + W^{1,s}(R^N) + BV(R^N)$ for some r and s such that $1 \leqslant r, s \leqslant \infty$ where $BV(R^N)$ denotes the space of L^1 functions whose first order distributional derivatives are bounded measures (the role of decompositions like $W^{1,r}(R^N) + W^{1,s}(R^N)$ is to cover potentials whose local Sobolev regularity is different from their Sobolev regularity at infinity while the role of $BV(R^N)$ is to take into account some discontinuous potentials). It is also possible to drop the condition $p < \frac{N}{N-2}$ by imposing suitable conditions on the potential; see Remark 32. Finally, in Section 6 we show that the semigroups $\{S_p(t); t \geqslant 0\}$ are holomorphic; this result is known if the relative bound of V is small; see e.g. [21,23,37], [33, p. 253].

Some spectral properties of the Schrödinger semigroups considered here and in [29] will be given in a forthcoming paper. The author thanks C. Villani and F. Murat for helpful remarks on the L^1-Laplacian.

2. Desch's theorem

The starting point is a remarkable perturbation result by W. Desch [13] which already has relevant applications to neutron transport with singular cross-sections (see [27, Chapter 9]).

Theorem 1. *(See [13, Desch's theorem].) Let X be an ordered Banach space such that the norm is additive on the positive cone X_+. Let $T : D(T) \subset X \to X$ be the generator of a positive c_0-semigroup $\{U(t); t \geqslant 0\}$ on X. Let $B : D(T) \subset X \to X$ be a positive operator, i.e.*

$$B : D(T) \cap X_+ \to X_+,$$

such that the resolvent $(\lambda - T - B)^{-1}$ exists for large λ and is positive (i.e. leaves invariant the positive cone). Then

$$T + B : D(T) \to X$$

generates a positive semigroup $\{V(t); t \geqslant 0\}$ on X.

The peculiarity of this result is that the mere existence (and positivity) of the resolvent of the perturbed operator $T + B$ is sufficient to assert that the latter is a generator. Desch's theorem [13] was given initially in AL-spaces, i.e. in Banach *lattices* with an additive norm on the positive cone. Actually, this theorem is true without the lattice assumption as it was pointed out in a remark of [4, p. 113]; a detailled proof of this is given in [28]. For more information on Desch's theorem and related topics we refer the reader to [27, Chapter 8] and [5, Chapter 5] where the following standard result can also be found.

Lemma 2. *Let X be an ordered Banach space with a generating and normal positive cone X_+. Let $T : D(T) \subset X \to X$ be a resolvent positive operator with spectral bound $s(T)$ and let $B : D(T) \subset X \to X$ be a positive operator (i.e. $B : D(T) \cap X_+ \to X_+$). Then, for $\lambda > s(T)$, the following assertions are equivalent:*

(i) $r_\sigma[B(\lambda - T)^{-1}] < 1$.
(ii) $\lambda \in \rho(T + B)$ *and* $(\lambda - T - B)^{-1} \geqslant 0$.

If one of these conditions is satisfied then $T + B$ is resolvent positive, $\lambda > s(T + B)$ and

$$(\lambda - T - B)^{-1} = (\lambda - T)^{-1} \sum_{j=0}^{\infty} \big[B(\lambda - T)^{-1}\big]^j \geqslant (\lambda - T)^{-1}.$$

3. (Weak) compactness and generation

We show here how (weak) compactness arguments provide useful generation results of perturbative type in ordered Banach spaces with additive norm on the positive cone. Before this, we give first a new result on the spectral radius of positive operators in general ordered Banach spaces which is needed in the sequel (this result is of course well known in Banach *lattices*). Let X be an ordered Banach space with norm $\| \ \|$ and positive cone X_+. We assume that the positive cone is generating, i.e. $X = X_+ - X_+$. We recall (see e.g. [6, Proposition 1.1.2]) that by a Baire category argument, there exists a constant $\gamma > 0$ such that each $x \in X$ has a decomposition $x = x_1 - x_2$; $x_1, x_2 \in X_+$ with

$$\|x_1\|, \|x_2\| \leqslant \gamma \|x\|. \tag{3}$$

For each positive bounded linear operator $C : X \to X$ we define

$$\|C\|_+ := \sup_{\|x\| \leqslant 1,\, x \in X_+} \|Cx\|.$$

We note that $\|Cx\| \leqslant \|C\|_+ \|x\|$, $\forall x \in X_+$ and $\|C_1 C_2\|_+ \leqslant \|C_1\|_+ \|C_2\|_+$ for positive operators C_1, C_2. Finally, $\|C\|_+ \leqslant \|C\|$ and this inequality might a priori be *strict* if X has not a lattice structure.

Lemma 3. *Let C be a positive bounded linear operator in an ordered Banach space with a generating cone. Then*

$$\lim_{n\to\infty} \left\|C^n\right\|_+^{\frac{1}{n}} = \inf_{n\geqslant 0} \left\|C^n\right\|_+^{\frac{1}{n}} \tag{4}$$

and

$$r_{\sigma+}(C) = r_\sigma(C), \tag{5}$$

where $r_{\sigma+}(C) := \lim_{n\to\infty} \|C^n\|_+^{\frac{1}{n}}$. Moreover, if X_+ is normal and if C_1 and C_2 are positive operators such that $C_1 \leqslant C_2$ then $r_\sigma(C_1) \leqslant r_\sigma(C_2)$.

Proof. The proof of (4) is the same as the standard one for a spectral radius (see e.g. [44, p. 212]) and is omitted. Let $x \in X$ be arbitrary and let a decomposition $x = x_1 - x_2$ with $x_1, x_2 \in X_+$ satisfying (3), then

$$\begin{aligned} \|Cx\| &= \|Cx_1 - Cx_2\| \leqslant \|Cx_1\| + \|Cx_2\| \leqslant \|C\|_+ \|x_1\| + \|C\|_+ \|x_2\| \\ &= \|C\|_+ \big[\|x_1\| + \|x_2\|\big] \leqslant 2\gamma \|C\|_+ \|x\| \end{aligned}$$

so $\|C\| \leqslant 2\gamma \|C\|_+$. Thus

$$\left\|C^n\right\|_+^{\frac{1}{n}} \leqslant \left\|C^n\right\|^{\frac{1}{n}} \leqslant (2\gamma)^{\frac{1}{n}} \left\|C^n\right\|_+^{\frac{1}{n}}$$

which ends the proof of (5) by letting $n \to \infty$. Finally, X_+ is normal if and only if there exists $\alpha \geqslant 1$ such that the norm is α-monotone, i.e. for $x, y \in X_+$ with $x \leqslant y$ we have $\|x\| \leqslant \alpha \|y\|$ (see e.g. [32, Proposition A.2.2, p. 266]). Let $C_1 \leqslant C_2$. Then $C_1^n x \leqslant C_2^n x$, $\forall x \in X_+$, $\forall n$ and

$$\|C_1^n\|_+^{\frac{1}{n}} \leqslant \alpha^{\frac{1}{n}} \|C_2^n\|_+^{\frac{1}{n}}, \quad \forall n,$$

so that $r_{\sigma+}(C_1) \leqslant r_{\sigma+}(C_2)$ and consequently (5) ends the proof. □

We note that according to Lemma 3, we can replace r_σ by $r_{\sigma+}$ in Lemma 2 so that (i) is satisfied once $\|B(\lambda - T)^{-1}\|_+ < 1$; this is useful when a lattice structure is lacking. We are now ready to show a basic result.

Theorem 4. *Let X be an ordered Banach space with a generating positive cone X_+ such that the norm is additive on X_+. Let $T : D(T) \subset X \to X$ be the generator of a positive c_0-semigroup on X and let $B : D(T) \to X$ be a linear positive operator. We assume that there exist $\bar{\lambda} > s(T)$ and an integer n such that $[B(\bar{\lambda} - T)^{-1}]^n$ is a compact operator. Then $T + B$ is a generator of a positive c_0-semigroup.*

Proof. Note that an additive norm on X_+ is monotone, i.e. if $x, y \in X_+$ and $x \leqslant y$ then $\|x\| \leqslant \|y\|$. This shows that for $\lambda \geqslant \bar{\lambda}$

$$\begin{aligned}\big\|\big[B(\lambda - T)^{-1}\big]^{n+1}\big\|_+ &= \big\|B(\lambda - T)^{-1}\big[B(\lambda - T)^{-1}\big]^n\big\|_+ \\ &\leqslant \big\|B(\lambda - T)^{-1}\big[B(\bar{\lambda} - T)^{-1}\big]^n\big\|_+.\end{aligned}$$

On the other hand,

$$\big\|B(\lambda - T)^{-1}\big[B(\bar{\lambda} - T)^{-1}\big]^n\big\|_+ \leqslant \big\|B(\lambda - T)^{-1}\big[B(\bar{\lambda} - T)^{-1}\big]^n\big\|.$$

Note that the positivity of $B : D(T) \to X$ implies that $B(\lambda - T)^{-1}$ is positive and therefore bounded. Thus B is T-bounded and

$$\big\|\big[B(\lambda - T)^{-1}\big]^{n+1}\big\|_+ \leqslant \|B\|_{L(D(T),X)} \big\|(\lambda - T)^{-1}\big[B(\bar{\lambda} - T)^{-1}\big]^n\big\|_{L(X,D(T))}.$$

Let $N(\cdot)$ be the T-graph norm. We note that $\|(\lambda - T)^{-1}x\| \to 0$ as $\lambda \to \infty$ and, for $x \in D(T)$, $\|T(\lambda - T)^{-1}x\| = \|(\lambda - T)^{-1}Tx\| \to 0$ as $\lambda \to \infty$. Since $T(\lambda - T)^{-1} = I + \lambda(\lambda - T)^{-1}$ is uniformly bounded for large λ then, for all $x \in X$, $\|T(\lambda - T)^{-1}x\| \to 0$ as $\lambda \to \infty$. Thus

$$\forall x \in X, \quad N\big((\lambda - T)^{-1}x\big) \to 0 \quad \text{as } \lambda \to \infty;$$

the convergence being uniform on compact subsets of X. Finally the compactness of $[B(\bar{\lambda} - T)^{-1}]^n$ shows that

$$\big\|(\lambda - T)^{-1}\big[B(\bar{\lambda} - T)^{-1}\big]^n\big\|_{L(X,D(T))} \to 0 \quad \text{as } \lambda \to \infty.$$

Thus $\|[B(\lambda - T)^{-1}]^{n+1}\|_+ \to 0$ as $\lambda \to \infty$ and consequently $r_{\sigma+}[B(\lambda - T)^{-1}] < 1$ for λ large enough by Lemma 3. Hence $T + B$ is resolvent positive by Lemma 2 and consequently Desch's theorem (Theorem 1) ends the proof. □

Remark 5. Theorem 4 is known in AL spaces with $n = 1$ [34, p. 19].

We provide now an important consequence of Theorem 4 for AL spaces.

Theorem 6. *Let X be an AL space and $T : D(T) \subset X \to X$ be the generator of a positive c_0-semigroup on X. Let $B : D(T) \subset X \to X$ be a positive weakly compact operator where $D(T)$ is endowed with the graph norm. Then $T + B$ is a generator of a positive c_0-semigroup.*

Proof. The weak compactness of $B : D(T) \subset X \to X$ amounts to the weak compactness of $B(\lambda - T)^{-1}$. Since the product of two weakly compact operators on an AL space is a compact operator [2, Corollary 19.9, p. 337] then $[B(\lambda - T)^{-1}]^2$ is compact and Theorem 4 ends the proof. □

We also give a useful (and simple) improvement of Theorem 6.

Corollary 7. *Let X be an AL space and $T : D(T) \subset X \to X$ be the generator of a positive c_0-semigroup on X. Let $B_i : D(T) \subset X \to X$ $(i = 1, 2)$ be two positive operators. We assume that $B_1 : D(T) \to X$ is weakly compact and $\lim_{\lambda\to\infty} \|B_2(\lambda - T)^{-1}\| < 1$. Then $T + B_1 + B_2$ is a generator of a positive c_0-semigroup.*

Proof. According to Lemma 2, $r_\sigma[B_2(\lambda - T)^{-1}] < 1$ for large λ so that $T + B_2 : D(T) \to X$ is resolvent positive. By Theorem 1, $T + B_2$ is a generator of a positive semigroup. Since B_1 is T-weakly compact, or equivalently $(T + B_2)$-weakly compact, then Theorem 6 ends the proof. □

4. On Schrödinger operators with negative potentials

Let $\{H_p(t); t \geqslant 0\}$ be the Heat semigroup on $L^p(R^N)$ $(1 \leqslant p \leqslant \infty)$

$$H_p(t)f = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{R^N} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy, \quad f \in L^p(R^N).$$

We denote by Δ_p its (Laplacian) generator with domain

$$D(\Delta_p) = \{\varphi \in L^p(R^N);\ \Delta\varphi \in L^p(R^N)\}.$$

The resolvent of the generator is given by

$$(\lambda - \Delta_p)^{-1} f = \int_0^{+\infty} e^{-\lambda t} H_p(t) f\,dt = \int_{R^N} G_\lambda(x - y) f(y)\,dy,$$

where G_λ is defined by $\widehat{G_\lambda}(\zeta) = \frac{(2\pi)^{-\frac{N}{2}}}{\lambda+|\zeta|^2}$. It is clear that $G_\lambda(\cdot)$ is C^∞ on $R^N - \{0\}$ and $G_\lambda(z) > 0$ for $z \neq 0$.

4.1. Generation results

Because of the positivity of V, it is an elementary fact that $V(\lambda - \Delta_1)^{-1}$ is a bounded operator on $L^1(R^N)$ for some (or equivalently all) $\lambda > 0$, i.e. V is Δ-bounded in $L^1(R^N)$, if and only if

$$x \to \int_{R^N} G_\lambda(x-y)V(y)\,dy \in L^\infty\big(R^N\big). \tag{6}$$

In such a case

$$\big\|V(\lambda - \Delta_1)^{-1}\big\|_{\mathcal{L}(L^1(R^N))} = \sup_{x\in R^N} \int_{R^N} G_\lambda(x-y)V(y)\,dy.$$

It is known [18,40] (see also [43, Proposition 5.1]) that (6) can be expressed in terms of a fundamental solution g_N of the Laplacian on R^N by

$$x \to \int_{|x-y|\leqslant 1} \big|g_N(x-y)\big|V(y)\,dy \in L^\infty\big(R^N\big). \tag{7}$$

In dimension $N = 1$, the class of potentials V satisfying (7) is called the Kato class K_1. In dimension $N \geqslant 2$, the Kato class, noted K_N, refers rather to the subclass of potentials such that

$$\lim_{\alpha\downarrow 0_+}\left(\operatorname{ess\,sup}_{x\in R^N} \int_{|x-y|\leqslant\alpha} \big|g_N(x-y)\big|V(y)\,dy\right) = 0. \tag{8}$$

In all this paper, the potential V is assumed to be Δ-bounded in $L^1(R^N)$, i.e. V is assumed to satisfy (7). Then it is well known (as a consequence of (6) for instance) that

$$V \in L^1_{\mathrm{loc}}\big(R^N\big).$$

We start with a basic observation:

Theorem 8. *Let* (7) *be satisfied. Then* $A_1 := \Delta_1 + V : D(\Delta_1) \to L^1(R^N)$ *is a generator of a positive semigroup* $\{S_1(t); t \geqslant 0\}$ *in* $L^1(R^N)$ *if and only if*

$$\lim_{\lambda\to+\infty} r_\sigma\big[V(\lambda - \Delta_1)^{-1}\big] < 1. \tag{9}$$

Proof. Note that $\lambda > 0 \to r_\sigma[V(\lambda - \Delta_1)^{-1}]$ in nonincreasing and then the limit (9) exists. If $r_\sigma[V(\lambda - \Delta_1)^{-1}] < 1$ for large λ then $(\lambda - \Delta_1 - V)^{-1}$ exists and is positive so that $A_1 := \Delta_1 + V : D(\Delta_1) \to L^1(R^N)$ is a generator of a positive semigroup $\{S_1(t); t \geqslant 0\}$ by Desch's theorem. Conversely, if $\Delta_1 + V : D(\Delta_1) \to L^1(R^N)$ is a generator of a positive semigroup then

$(\lambda - \Delta_1 - V)^{-1}$ exists and is positive for large λ and, by Lemma 2, $r_\sigma[V(\lambda - \Delta_1)^{-1}] < 1$ for large λ. □

Remark 9. One sees that for Δ-bounded potentials in L^1 sense, assumption (9) is optimal for the L^1 generation theory.

Remark 10. It is well known (see [1, Theorem 4.14], [40, Proposition A.2.3], [21, Lemma 11]) that the parameter

$$c_N(V) := \lim_{\lambda\to\infty} \left\| V(\lambda - \Delta_1)^{-1} \right\|_{\mathcal{L}(L^1(R^N))} \tag{10}$$

provides the relative bound of V with respect to Δ_1 and this limit can be "computed"

$$c_N(V) = \lim_{\alpha\downarrow 0_+} \left(\operatorname{ess\,sup}_{x\in R^N} \int_{|x-y|\leqslant\alpha} \left| g_N(x-y) \right| V(y)\, dy \right),$$

i.e. the class of potentials V satisfying $c_N(V) = 0$ is the Kato class. The weaker assumption $c_N(V) < 1$ appears for instance in [14,24,43]. Actually, Theorem 8 suggests to consider rather $\lim_{\lambda\to+\infty} r_\sigma[V(\lambda - \Delta_1)^{-1}]$ as the relevant parameter; see Section 5 below for more information.

Remark 11. Note that a priori $\| V_1(\lambda - \Delta_1)^{-1} \|_{\mathcal{L}(L^1(R^N))}$ need not go to 0 as $\lambda \to +\infty$ so that the potential V_1 does not belong a priori to the Kato class. Further, a priori we may have $c_N(V_1) > 1$ without preventing the generation property; see Remark 16. A similar phenomenon arises in neutron transport theory; see [27, Chapter 9].

Remark 12. We can of course add a bounded potential to the generator $\Delta_1 + V$ without changing the conclusions in Theorem 8. We can also add a negative term $\widetilde{V} \in L^1_{\mathrm{loc}}$ to V without changing the conclusions in Theorem 8. Indeed, $\widetilde{V}$ acts as an absorption term which decreases the resolvent, i.e.

$$\left(\lambda - (\Delta_1 + \widetilde{V})\right)^{-1} \leqslant (\lambda - \Delta_1)^{-1},$$

so that

$$r_\sigma\left[V\left(\lambda - (\Delta_1 + \widetilde{V})\right)^{-1}\right] \leqslant r_\sigma\left[V(\lambda - \Delta_1)^{-1}\right] < 1$$

for large λ. Actually, the meaning of $\Delta_1 + V + \widetilde{V}$ is "$(\Delta_1 + \widetilde{V})$" $+ V$ on the domain of "$(\Delta_1 + \widetilde{V})$" where "$(\Delta_1 + \widetilde{V})$" is defined by a suitable truncation and monotonic limiting procedure. We refer to [3,43] and references therein for a systematic treatment of absorption semigroups.

Theorems 13, 15 and 17 below provide concrete realizations of Theorems 4 and 8.

Theorem 13. *Let* (7) *be satisfied. We assume that there exists an integer k such that $[V(\lambda - \Delta_1)^{-1}]^k$ is a compact operator. Then $A_1 : \Delta_1 + V : D(\Delta_1) \to L^1(R^N)$ is a generator of a positive semigroup $\{S_1(t); t \geqslant 0\}$ in $L^1(R^N)$.*

Proof. This is simply a particular case of (the abstract) Theorem 4. □

Remark 14. Note that for $k=1$ we find again a known result corresponding to $c_N(V)=0$; see e.g. [40, Proposition A.2.3].

Theorem 15. *Let $V=V_1+V_2$ with $V_1, V_2 \geqslant 0$. Let* (7) *be satisfied and $c_N(V_2)<1$. If V_1 is such that the* (*bounded*) *subset of $L^1(R^N)$*

$$\left\{G_1(x-\cdot)V_1(\cdot);\ x \in R^N\right\}$$

is equi-integrable then $A_1 : \Delta_1 + V : D(\Delta_1) \to L^1(R^N)$ is a generator of a positive semigroup $\{S_1(t); t \geqslant 0\}$ in $L^1(R^N)$.

Proof. Note that $c_N(V_2)<1$ amounts to $\|V_2(\lambda-\Delta_1)^{-1}\|_{\mathcal{L}(L^1(R^N))}<1$ for large λ. On the other hand, according to Corollary 7, it suffices to show that $V_1(\lambda-\Delta_1)^{-1} : L^1(R^N) \to L^1(R^N)$ is weakly compact, i.e.

$$\left\{V_1(\lambda-\Delta_1)^{-1}f;\ \|f\|_{L^1(R^N)} \leqslant 1\right\}$$

is equi-integrable. According to the general criteria of equi-integrability (see e.g. [15]), this is equivalent to

$$\int_{\Omega} \left|V_1(x)\big((\lambda-\Delta_1)^{-1}f\big)(x)\right| dx \to 0 \quad \text{as } |\Omega| \to 0$$

($|\Omega|$ is the Lebesgue measure of Ω) uniformly in f in the unit ball of $L^1(R^N)$ and

$$\int_{E_j^c} \left|V_1(x)\big((\lambda-\Delta_1)^{-1}f\big)(x)\right| dx \to 0$$

uniformly in f in the unit ball of $L^1(R^N)$ as $j\to\infty$ for any increasing sequence $\{E_j\}$ of measurable subsets of R^N (with finite measure) such that $\bigcup E_j = R^N$. Actually, we can restrict ourselves to nonnegative f. Thus, the estimate

$$\int_{\Omega} V_1(x)\big((\lambda-\Delta_1)^{-1}f\big)(x)\, dx = \int_{R^N} f(y)\left[\int_{\Omega} G_\lambda(x-y)V_1(x)\, dx\right] dy$$

shows that

$$\sup_{\|f\|_{L^1_+(R^N)} \leqslant 1} \int_{\Omega} V_1(x)\big((\lambda-\Delta_1)^{-1}f\big)(x)\, dx = \operatorname{ess\,sup}_{y\in R^N} \int_{\Omega} G_1(x-y)V_1(x)\, dx.$$

Similarly

$$\sup_{\|f\|_{L^1_+(R^N)}\leqslant 1}\int_{E_j^c} V_1(x)\big((\lambda-\Delta_1)^{-1}f\big)(x)\,dx = \operatorname{ess}\sup_{y\in R^N}\int_{E_j^c} G_1(x-y)V_1(x)\,dx$$

and consequently the weak compactness of $V_1(\lambda-\Delta_1)^{-1}$ is equivalent to our equi-integrability assumption. □

Remark 16. We note that the size of the set $\{G_1(x-\cdot)V_1(\cdot);\, x\in R^N\}$ is nothing but

$$\sup_{x\in R^N}\int G_1(x-y)V_1(y)\,dy = \big\|V_1(1-\Delta_1)^{-1}\big\|_{\mathcal{L}(L^1(R^N))}.$$

Thus, Theorem 15 shows that under an equi-integrability assumption, the size of the set $\{G_1(x-\cdot)V_1(\cdot);\, x\in R^N\}$ is irrelevant and, a priori, $c_N(V_1)$ need not be small.

We give a slightly different version of Theorem 15 which separates the local role of V_1 from its role at infinity.

Theorem 17. *Let $V=V_1+V_2$ $(V_1,V_2\geqslant 0)$ satisfying (7) and let $c_N(V_2)<1$. Let*

$$\int_{|x|\geqslant c} G_1(x-y)V_1(x)\,dx\to 0 \quad \text{as } c\to\infty \tag{11}$$

uniformly in $y\in R^N$. We assume that

$$V_1(1-\Delta_1)^{-1}: L^1\big(R^N\big)\to L^1_{\mathrm{loc}}\big(R^N\big) \text{ is weakly compact.} \tag{12}$$

Then $A_1:\Delta_1+V: D(\Delta_1)\to L^1(R^N)$ is a generator of a positive semigroup $\{S_1(t);\, t\geqslant 0\}$ in $L^1(R^N)$. Moreover, (11) is satisfied if for c large enough, the function

$$F_c: y\in R^N\to \int_{|x|\geqslant c} G_1(x-y)V_1(x)\,dx$$

is continuous and goes to zero as $|y|\to\infty$.

Proof. We have

$$\begin{aligned}
\int_{|x|\geqslant c}\big|\big[V_1(1-\Delta_1)^{-1}f\big]\big|\,dx &\leqslant \int_{|x|\geqslant c} V_1(x)\Big[\int_{R^N} G_1(x-y)\big|f(y)\big|\,dy\Big]dx\\
&= \int_{R^N}\big|f(y)\big|\Big[\int_{|x|\geqslant c} G_1(x-y)V_1(x)\,dx\Big]\\
&\leqslant \sup_{y\in R^N}\Big[\int_{|x|\geqslant c} G_1(x-y)V_1(x)\,dx\Big]\|f\|_{L^1}
\end{aligned}$$

so that

$$\int_{|x|\geqslant c} \left|\left[V_1(1-\Delta_1)^{-1}f\right]\right| dx \to 0 \quad \text{as } c \to \infty$$

uniformly in $\|f\|_{L^1} \leqslant 1$. This shows that

$$\chi_{\{|x|<c\}} V_1(1-\Delta_1)^{-1} \to V_1(1-\Delta_1)^{-1} \quad \text{as } c \to \infty$$

in operator norm. By assumption $\chi_{\{|x|<c\}} V_1(1-\Delta_1)^{-1}$ is weakly compact whence $V_1(1-\Delta_1)^{-1}$ is also weakly compact and then the generation property holds. To show the last statement, it suffices to show that for some increasing sequence $c_j \to +\infty$, $F_{c_j}(y) \to 0$ as $j \to \infty$ uniformly in $y \in R^N$. We note that by assumption $\{F_{c_j}\}_j$ is a decreasing sequence of continuous functions and for each $y \in R^N$, $F_{c_j}(y) \to 0$. By Dini's theorem the convergence is uniform in y on any compact subset of R^N. On the other hand, for any $\varepsilon > 0$ there exists a constant $\gamma > 0$ such that $F_{c_0}(y) \leqslant \varepsilon$ for $|y| > \gamma$ and consequently $F_{c_j}(y) \leqslant F_{c_0}(y) \leqslant \varepsilon$ for $|y| > \gamma$ for all j. This shows that $F_{c_j}(y) \to 0$ uniformly in $y \in R^N$ as $j \to \infty$. □

Remark 18. Assumption (11) expresses that the potential V_1 is "small at infinity" in some averaged sense:

(i) For instance, for $N \geqslant 2$, (11) is satisfied if $V_1 \in L^q(B^{\text{ext}})$ for some $q \in]\frac{N}{2}, \infty[$ where B^{ext} is the *exterior* of $B := \{x; |x| \leqslant c\}$ (or if $V_1(x) \to 0$ as $|x| \to \infty$). Indeed, define V_c by: $V_c = 0$ on B and $V_c = V_1$ on B^{ext}. Then $F_c = G_\lambda * V_c \in \mathcal{C}_0(R^N)$ since $V_c \in L^q(R^N)$ for some $q \in]\frac{N}{2}, \infty[$ and $G_\lambda(\cdot) \in L^s(R^N)$ for all $s \in [1, \frac{N}{N-2}[$ (see [45, p. 65]) so that we can choose $s = q^*$ the conjugate exponent of q. On the other hand, if $V_1(x) \to 0$ as $|x| \to \infty$, one sees directly that

$$\int_{|x|\geqslant c} G_1(x-y)V_1(x)\,dx \leqslant \|G_1\|_{L^1} \sup_{|x|\geqslant c} V_1(x) \to 0 \quad \text{as } c \to \infty.$$

(ii) Assumption (11) can also be checked by noting that $G_\lambda * V_c \in \mathcal{C}_0(R^N)$ if V_c is a tempered distribution such that $\widehat{G_\lambda}(\zeta)\widehat{V_c}(\zeta) \in L^1(R^N)$, i.e. $\frac{\widehat{V_c}(\zeta)}{\lambda+|\zeta|^2} \in L^1(R^N)$. This condition on the potential is a priori different from that given in (i).

(iii) We note that in dimension $N = 1$, (12) is always satisfied; actually $V(1-\Delta_1)^{-1} : L^1(R) \to L^1_{\text{loc}}(R)$ is compact because $(1-\Delta_1)^{-1}$ maps continuously $L^1(R)$ into $W^{2,1}(R)$, the restriction to a bounded interval $[a, b] \subset R$ of a bounded set of $W^{2,1}(R)$ is relatively compact in $C([a, b])$ (equipped with the supremum norm) and $V \in L^1_{\text{loc}}(R)$. Thus, for $N = 1$, V is Δ_1-compact if (11) is satisfied, e.g. if there exists $q \in [1, +\infty[$ such that $(V)^q$ is integrable at infinity, or if $V(x) \to 0$ as $|x| \to \infty$.

We show now how to check assumption (12) in dimension $N \geqslant 2$.

Proposition 19. *We assume that for any ball B centered at zero and any $c > 0$*

$$\lim_{|\Omega|\to 0,\, \Omega\subset B} \sup_{|x|\leqslant c} \int_{\Omega} \left|g_N(x-y)\right| V_1(y)\, dy = 0, \tag{13}$$

where $|\Omega|$ is the Lebesgue measure of Ω. Then

$$V_1(1-\Delta_1)^{-1} : L^1\big(R^N\big) \to L^1_{\mathrm{loc}}\big(R^N\big)$$

is weakly compact.

Proof. Let $b > 0$ be a constant and B be the ball with radius b and centered at the origin. Let us show that $V_1(1-\Delta_1)^{-1} : L^1(R^N) \to L^1(B)$ is weakly compact. By the Dunford–Pettis criterion, we have to check that

$$\int_{\Omega} \left|V_1(1-\Delta_1)^{-1} f\right| dx \to 0$$

as $|\Omega| \to 0$ ($\Omega \subset B$) uniformly in $\|f\|_{L^1(R^N)} \leqslant 1$. Let $\delta > 0$ be fixed. We note that

$$\begin{aligned}
\int_{\Omega} \left|V_1(1-\Delta_1)^{-1} f\right| dx &\leqslant \int_{\Omega} V_1(x) \left[\int_{R^N} G_1(x-y)\left|f(y)\right| dy\right] dx \\
&= \int_{R^N} \left|f(y)\right| \left[\int_{\Omega} G_1(x-y) V_1(x)\, dx\right] dy \\
&= \int_{|y|\geqslant b+\delta} \left|f(y)\right| \left[\int_{\Omega} G_1(x-y) V_1(x)\, dx\right] dy \\
&\quad + \int_{|y|< b+\delta} \left|f(y)\right| \left[\int_{\Omega} G_1(x-y) V_1(x)\, dx\right] dy.
\end{aligned} \tag{14}$$

There exists c_δ such that $G_1(x-y) \leqslant c_\delta$ if $|x-y| \geqslant \delta$ so that

$$\begin{aligned}
\int_{|y|\geqslant b+\delta} \left|f(y)\right| \left[\int_{\Omega} G_1(x-y) V_1(x)\, dx\right] dy &\leqslant c_\delta \int_{|y|\geqslant b+\delta} \left|f(y)\right| \left[\int_{\Omega} V_1(x)\, dx\right] dy \\
&\leqslant c_\delta \|f\|_{L^1(R^N)} \int_{\Omega} V_1(x)\, dx
\end{aligned}$$

because $\Omega \subset B$. Thus $\int_{|y|\geqslant b+\delta} |f(y)|[\int_{\Omega} G_1(x-y) V_1(x)\, dx]\, dy \to 0$ as $|\Omega| \to 0$ uniformly in $\|f\|_{L^1(R^N)} \leqslant 1$. To deal with the last term in (14), we note that there exists $c' > 0$ such that $G_1(x-y) \leqslant c'|g_N(x-y)|$, $\forall x, y$ so that

$$\int_{|y|<b+\delta} |f(y)| \left[\int_{\Omega} G_1(x-y) V_1(x)\, dx \right] dy \leqslant c' \int_{|y|<b+\delta} |f(y)| \left[\int_{\Omega} |g_N(x-y)| V_1(x)\, dx \right] dy$$

$$\leqslant c' \sup_{|y|<b+\delta} \int_{\Omega} |g_N(x-y)| V_1(x)\, dx \|f\|_{L^1(R^N)} \to 0$$

as $|\Omega| \to 0$ uniformly in $\|f\|_{L^1(R^N)} \leqslant 1$ thanks to assumption (13). □

Remark 20. We recall that the local class K_N^{loc} (given in [1, p. 210]) refers to the potentials V such that for any $c > 0$

$$\lim_{\alpha \downarrow 0_+} \left(\sup_{|x| \leqslant c} \int_{|x-y| \leqslant \alpha} |g_N(x-y)| V(y)\, dy \right) = 0.$$

We observe that our assumption (13) is weaker than requiring $V_1 \in K_N^{\mathrm{loc}}$. Indeed, let $V_1 \in K_N^{\mathrm{loc}}$. Then

$$\begin{aligned} \int_{\Omega} |g_N(x-y)| V_1(y)\, dy &= \int_{\Omega \cap \{|x-y| \leqslant \alpha\}} |g_N(x-y)| V_1(y)\, dy \\ &\quad + \int_{\Omega \cap \{|x-y| \geqslant \alpha\}} |g_N(x-y)| V_1(y)\, dy \\ &\leqslant \sup_{|x| \leqslant c} \int_{|x-y| \leqslant \alpha} |g_N(x-y)| V_1(y)\, dy + c' \int_{\Omega} V_1(y)\, dy, \end{aligned} \tag{15}$$

where $c' := \sup_{|x| \leqslant c,\, y \in B,\, |x-y| \geqslant \alpha} |g_N(x-y)|$. Let $\varepsilon > 0$ be arbitrary. Under the assumption $V_1 \in K_N^{\mathrm{loc}}$ we have

$$\sup_{|x| \leqslant c} \int_{|x-y| \leqslant \alpha} |g_N(x-y)| V_1(y)\, dy \leqslant \varepsilon$$

for α small enough and then, for α fixed, the term in (15) goes to zero as $|\Omega| \to 0$. Thus V_1 satisfies (13).

There is also another way to see why assumption (13) is weaker than requiring $V_1 \in K_N^{\mathrm{loc}}$: It is known (see [40, Proposition A.2.4]) that $V_1 \in K_N^{\mathrm{loc}}$ is equivalent to the compactness of $(1 - \Delta_\infty)^{-1}(\chi_B V_1) : L^\infty(R^N) \to L^\infty(R^N)$. Thus, by a duality argument, $V_1 \in K_N^{\mathrm{loc}}$ is equivalent to the compactness of the operator $V_1(1 - \Delta_1)^{-1} : L^1(R^N) \to L^1_{\mathrm{loc}}(R^N)$ while Proposition 19 deals rather with the weak compactness of this operator.

Remark 21. We note that $V_1 \in K_N^{\mathrm{loc}}$ combined with (11) would imply that $V_1(1 - \Delta_1)^{-1}$: $L^1(R^N) \to L^1(R^N)$ is compact, i.e. V_1 is Δ_1-compact and then $c_N(V_1) = 0$.

The following lemma is crucial to deal with L^p spaces.

Lemma 22. *Under the general assumption of Theorem* 8 *and for* λ *large enough* $(\lambda - \Delta_1 - V)^{-1}$ *and* $S_1(t)$ *are symmetric operators, i.e. for all* $g_1, g_2 \in L^1(R^N) \cap L^\infty(R^N)$

$$\int \big((\lambda - \Delta_1 - V)^{-1} g_1\big) g_2 \, dx = \int g_1 \big((\lambda - \Delta_1 - V)^{-1} g_2\big) \, dx$$

and

$$\int \big(S_1(t) g_1\big) g_2 \, dx = \int g_1 \big(S_1(t) g_2\big) \, dx. \tag{16}$$

Proof. We note that the problem

$$\lambda f - \Delta_1 f - V f = g$$

is equivalent to

$$f - (\lambda - \Delta_1)^{-1} V f = (\lambda - \Delta_1)^{-1} g.$$

The change of function $\widetilde{f} = V f$ leads to

$$f = (\lambda - \Delta_1)^{-1} \widetilde{f} + (\lambda - \Delta_1)^{-1} g$$

and

$$\widetilde{f} - V(\lambda - \Delta_1)^{-1} \widetilde{f} = V(\lambda - \Delta_1)^{-1} g.$$

By assumption $r_\sigma[V_2(\lambda - \Delta_1)^{-1}] < 1$ for large λ and

$$\widetilde{f} = \sum_{n=0}^{\infty} \big[V(\lambda - \Delta_1)^{-1}\big]^n V(\lambda - \Delta_1)^{-1} g \tag{17}$$

whence

$$\begin{aligned}
f &= (\lambda - \Delta_1)^{-1} \sum_{n=0}^{\infty} \big[V(\lambda - \Delta_1)^{-1}\big]^n V(\lambda - \Delta_1)^{-1} g + (\lambda - \Delta_1)^{-1} g \\
&= (\lambda - \Delta_1)^{-1} g + \sum_{n=0}^{\infty} (\lambda - \Delta_1)^{-1} \big[V(\lambda - \Delta_1)^{-1}\big]^{n+1} g \\
&= (\lambda - \Delta_1)^{-1} g + \sum_{n=1}^{\infty} (\lambda - \Delta_1)^{-1} \big[V(\lambda - \Delta_1)^{-1}\big]^{n} g \\
&= \sum_{n=0}^{\infty} (\lambda - \Delta_1)^{-1} \big[V(\lambda - \Delta_1)^{-1}\big]^{n} g
\end{aligned}$$

and

$$(\lambda - \Delta_1 - V)^{-1} = \sum_{n=0}^{\infty} (\lambda - \Delta_1)^{-1} \big[V (\lambda - \Delta_1)^{-1} \big]^n.$$

Clearly, each term of the series is an integral operator with nonnegative kernel. Moreover, if $f, g \in L^1(R^N) \cap L^\infty(R^N)$ then, using the symmetry of the integral operator $(\lambda - \Delta_1)^{-1}$, one easily sees that

$$\begin{aligned} \big\langle (\lambda - \Delta_1)^{-1} \big(V (\lambda - \Delta_1)^{-1} \big)^n f, g \big\rangle_{L^1, L^\infty} &= \big\langle f, \big((\lambda - \Delta_1)^{-1} V \big)^n (\lambda - \Delta_1)^{-1} g \big\rangle_{L^1, L^\infty} \\ &= \big\langle f, (\lambda - \Delta_1)^{-1} \big(V (\lambda - \Delta_1)^{-1} \big)^n g \big\rangle_{L^1, L^\infty} \end{aligned}$$

showing thus the symmetry of the integral operator $(\lambda - \Delta_1 - V)^{-1}$. Finally the exponential formula shows (16). □

Theorem 23. *Let the conditions in Theorem* 8 *be satisfied and let* $p \in]1, +\infty[$. *Let*

$$\Xi_p := \big\{ f \in L^p\big(R^N\big) \cap D(\Delta_1);\ \Delta f + V f \in L^p\big(R^N\big) \big\}.$$

Then

$$\Delta + V : \Xi_p \to L^p\big(R^N\big)$$

is closable and its closure A_p *generates a positive* c_0*-semigroup* $\{S_p(t); t \geqslant 0\}$.

Proof. It follows from Lemma 22 that the dual operator $[\lambda - (\Delta_1 - V)']^{-1}$ in $L^\infty(R^N)$ coincides with $(\lambda - \Delta_1 - V)^{-1}$ on $L^1(R^N) \cap L^\infty(R^N)$ for large λ (more precisely for λ such that $r_\sigma[V(\lambda - \Delta_1)^{-1}] < 1$). Similarly the dual semigroup $V_1'(t)$ in $L^\infty(R^N)$ coincides with $V_1(t)$ on $L^1(R^N) \cap L^\infty(R^N)$. It follows, by Riesz–Thorin interpolation theorem, that

$$V_1(t) : L^1\big(R^N\big) \cap L^p\big(R^N\big) \to L^1\big(R^N\big) \cap L^p\big(R^N\big)$$

extends uniquely to $L^p(R^N)$ as a bounded operator $V_p(t)$ on $L^p(R^N)$ for $p \in [1, +\infty]$. Clearly the semigroup property is preserved. Finally, the strong continuity in $L^p(R^N)$ for $p < \infty$ is inherited from the case $p = 1$. Thus $\{V_p(t); t \geqslant 0\}$ is a c_0-semigroup on $L^p(R^N)$. We denote by A_p its infinitesimal generator. Similarly $(\lambda - \Delta_1 - V)^{-1}$ extends uniquely to $L^p(R^N)$ as a bounded operator B_p on $L^p(R^N)$. The fact that

$$(\lambda - \Delta_1 - V)^{-1} f = \int_0^\infty e^{-\lambda t} \big[V_1(t) f \big] \, dt, \quad f \in L^1\big(R^N\big) \cap L^p\big(R^N\big)$$

shows that the resolvent $(\lambda - A_p)^{-1} : L^p(R^N) \to L^p(R^N)$ is nothing but B_p the unique extension to $L^p(R^N)$ of

$$(\lambda - \Delta_1 - V)^{-1} : L^1\big(R^N\big) \cap L^p\big(R^N\big) \to L^1\big(R^N\big) \cap L^p\big(R^N\big).$$

There exists a constant $c > 0$ such that

$$\left\|(\lambda - \Delta_1 - V)^{-1} g\right\|_{L^p} \leqslant c\|g\|_{L^p}, \quad g \in L^1\left(R^N\right) \cap L^p\left(R^N\right).$$

Let $f \in D(A_p)$. There exists $g \in L^p(R^N)$ such that $f = (\lambda - A_p)^{-1} g$. For any sequence $(g_k) \subset L^1(R^N) \cap L^p(R^N)$ such that $g_k \to g$ in $L^p(R^N)$ we have $f_k := (\lambda - \Delta_1 - V)^{-1} g_k \in L^p(R^N) \cap D(\Delta_1)$ and

$$f_k \to f := (\lambda - A_p)^{-1} g \quad \text{in } L^p\left(R^N\right).$$

Thus

$$\lambda f_k - \Delta_1 f_k - V f_k \to \lambda f - A_p f \quad \text{in } L^p\left(R^N\right),$$

$\Delta_1 f_k + V f_k \in L^p(R^N)$ and $\Delta_1 f_k + V f_k \to A_p f$ in $L^p(R^N)$, i.e. $(f_k) \subset \Xi$ and $(f_k, \Delta_1 f_k + V f_k) \to (f, A_p f)$ in $L^p(R^N) \times L^p(R^N)$. □

Corollary 24. *Let the conditions in Theorem* 8 *be satisfied. Then the closure of* $\Delta + V : \Xi_2 \to L^2(R^N)$ *is self-adjoint and semi-bounded.*

Proof. We know that A_2, the closure of $\Delta + V : \Xi_2 \to L^2(R^N)$, generates a self-adjoint semigroup. □

Remark 25. Theorem 8 could be stated in abstract $L^1(\mu)$ spaces (with a σ-finite measure μ) where the Heat semigroup is replaced by any positive semigroup $\{\widetilde{H}_1(t); t \geqslant 0\}$ having the symmetry property

$$\int \left(\widetilde{H}_1(t) g_1\right) g_2 \, dx = \int g_1 \left(\widetilde{H}_1(t) g_2\right) dx, \quad \forall g_1, g_2 \in L^1(\mu) \cap L^\infty(\mu).$$

Then the perturbed semigroup $\{\widetilde{S}_1(t); t \geqslant 0\}$ will inherit this symmetry and interpolates on $L^p(\mu)$ spaces. We can then derive an abstract version of Theorem 23. For instance, we could replace the Laplacian operator by a general symmetric second order elliptic operator with variable coefficients and with suitable boundary conditions in a domain $\Omega \subset R^N$.

4.2. On the domain generator

This subsection provides some additional information related to whether $\mathcal{C}_c^\infty(R^N)$ is a core for A_p. (Other results are given in Section 5 below.) We start with the case $p = 1$.

Theorem 26. *Under the general assumptions of Theorem* 8, $\mathcal{C}_c^\infty(R^N)$ *is a core for* A_1.

Proof. Note that $D(A_1) = D(\Delta_1)$ and $D(\Delta_1) \subset W^{1,1}(R^N)$ (see e.g. [45, p. 65]). Let $f \in D(A_1)$; then $V f \in L^1(R^N)$ since V is Δ_1-bounded. Let $\phi \in \mathcal{C}_c^\infty(R^N)$ with $\phi(0) = 1$ and let $\phi_n(x) = \phi(\frac{x}{n})$. Then $f_n := f\phi_n \in D(A_1)$, $f_n \to f$ in $L^1(R^N)$, $V f_n = (V f)\phi_n \to V f$ in $L^1(R^N)$ and

$$\Delta f_n = (\Delta\phi_n) f + 2\nabla\phi_n.\nabla f + (\Delta f)\phi_n \to \Delta f \quad \text{in } L^1\big(R^N\big).$$

Then the elements of $D(A_1)$ with compact supports form a core of A_1. Now, let $f \in D(A_1)$ with compact support and let $f_n := f * g_n$ where $(g_n)_n$ is a standard mollifier sequence. Then $f_n \in \mathcal{C}_c^\infty(R^N)$, $f_n \to f$ in $L^1(R^N)$ and

$$\Delta f_n = \Delta(f * g_n) = \Delta(f) * g_n \to \Delta(f) \quad \text{in } L^1\big(R^N\big)$$

so that $f_n \to f$ in the Δ_1-graph norm; it follows that $Vf_n \to Vf$ in $L^1(R^N)$ since V is Δ_1-bounded. Thus $f_n \in \mathcal{C}_c^\infty(R^N)$, $f_n \to f$ in $L^1(R^N)$ and $A_1 f_n \to A_1 f$ in $L^1(R^N)$. $\square$

Remark 27. The proof above is partly inspired by similar ideas scattered in the literature (e.g. [40, Theorem B.1.5] or [43, Theorem 7]). Thus, Theorem 26 is given in [43] under the assumption $c_N(V) < 1$ and its proof uses also other technical results from [14] which are unnecessary here.

According to a classical result by T. Kato [18], $\mathcal{C}_c^\infty(R^N)$ is a core for A_2 if $V \in L^2_{\mathrm{loc}}(R^N)$ and $c_N(V) = 0$. Actually, it was shown later [25] that for $p > 1$ and $V \in L^p_{\mathrm{loc}}(R^N)$, $\mathcal{C}_c^\infty(R^N)$ is a core for A_p provided that $c_N(V) < 1$. We treat here our general case differently under a technical assumption (which is always true for $p = 2$; see Section 5) we discuss in Corollary 31 below for general p. The general strategy of the proof is similar to that of Theorem 26 but combines additional technical arguments.

Theorem 28. *Let the general assumptions of Theorem* 8 *be satisfied. Let* $p > 1$ *and* $V \in L^p_{\mathrm{loc}}(R^N)$. *We assume that for each* $g \in \mathcal{S}(R^N)$ (*the Schwartz space*) *the solution* f *to the problem*

$$\lambda f - \Delta f - Vf = g, \quad f \in D(A_1) \tag{18}$$

(*which exists for* λ *large enough in all* L^q *spaces*) *has a gradient* $\nabla f \in L^p(R^N)$. *Then* $\mathcal{C}_c^\infty(R^N)$ *is a core for* A_p.

Proof. We observe first that since g is bounded then the solution f in (18) is also bounded. Indeed, this is a consequence of the fact, noted in the proof of Theorem 23, that $[\lambda - (\Delta_1 - V)']^{-1}$ in $L^\infty(R^N)$ coincides with $(\lambda - \Delta_1 - V)^{-1}$ on $L^1(R^N) \cap L^\infty(R^N)$. Let $\lambda > 0$ be large enough, i.e. λ such that $r_\sigma[V(\lambda - \Delta_1)^{-1}] < 1$. We already know that

$$\Xi_p := \big\{ f \in L^p\big(R^N\big) \cap D(\Delta_1);\ \Delta f + Vf \in L^p\big(R^N\big)\big\}$$

is a core of A_p. Actually, by inspecting the proof of Theorem 23, one sees that the sequence $(g_k) \subset L^1(R^N) \cap L^p(R^N)$ which approximates g in $L^p(R^N)$ is arbitrary. Therefore we can choose (g_k) in the Schwartz space $\mathcal{S}(R^N)$ so that (according to our extra assumption) we may replace Ξ_p by another core

$$\widetilde{\Xi}_p := \big\{ f \in L^\infty\big(R^N\big) \cap W^{1,p}\big(R^N\big) \cap D(\Delta_1);\ \Delta f + Vf \in L^p\big(R^N\big)\big\}.$$

Let $f \in \widetilde{\Xi}_p$. Let $\phi \in \mathcal{C}_c^\infty(R^N)$ with $\phi(0) = 1$, $\phi_n(x) = \phi(\frac{x}{n})$ and $f_n := f\phi_n$. Clearly $f_n \in L^p(R^N)$ and $f_n \to f$ in $L^p(R^N)$. Moreover (see the proof of Theorem 26) $f_n \in D(\Delta_1)$. On the other hand

$$\begin{aligned}\Delta f_n + V f_n &= (\Delta\phi_n) f + 2\nabla\phi_n.\nabla f + (\Delta f)\phi_n + V f_n\\ &= (\Delta\phi_n) f + 2\nabla\phi_n.\nabla f + [\Delta f + V f]\phi_n \in L^p\big(R^N\big)\end{aligned}$$

because $\nabla f \in [L^p(R^N)]^N$ and $\Delta f + V f \in L^p(R^N)$. Thus $(f_n) \subset \widetilde{\Xi}_p$. Moreover

$$\Delta f_n + V f_n = (\Delta\phi_n) f + 2\nabla\phi_n.\nabla f + [\Delta f + V f]\phi_n \to \Delta f + V f \text{ in } L^p\big(R^N\big).$$

Hence the elements of $\widetilde{\Xi}_p$ with compact supports form a core for A_p. Let now $f \in \widetilde{\Xi}_p$ with a compact support and let $f_n := f * g_n$ where $(g_n)_n$ is a standard mollifier sequence. Then $f_n \in \mathcal{C}_c^\infty(R^N)$, $f_n \to f$ in $L^p(R^N)$. We have also

$$\Delta f_n + V f_n = \Delta(f) * g_n + V(f * g_n) = \big[\Delta(f) + V f\big] * g_n + V(f * g_n) - (V f) * g_n.$$

Note that $\Delta(f) + V f \in L^p(R^N)$ and then $[\Delta(f) + V f] * g_n \to \Delta(f) + V f$ in $L^p(R^N)$. On the other hand

$$V(f * g_n) - (V f) * g_n = V[f * g_n - f] + \big[V f - (V f) * g_n\big].$$

Note that $f \in L^\infty(R^N)$ and f is compactly supported so that $V f \in L^p(R^N)$ because $V \in L^p_{\mathrm{loc}}(R^N)$ and then $V f - (V f) * g_n \to 0$ in $L^p(R^N)$. On the other hand, since f is bounded then $f * g_n - f$ is uniformly bounded too and its support is included in a bounded set independent of n. Moreover, since $f * g_n \to f$ in all L^q spaces with $q < \infty$ then, by extracting a subsequence if necessary, we can assume that $f * g_n - f \to 0$ a.e. and then, by the dominated convergence theorem, $V[f * g_n - f] \to 0$ in $L^p(R^N)$. Finally

$$\Delta f_n + V f_n \to \Delta(f) + V f \text{ in } L^p\big(R^N\big)$$

and we are done. □

Remark 29. Note that, in dimension $N = 1$, the solution f to (18) belongs to $W^{2,1}(R)$ so that the assumption concerning (18) is automatically satisfied.

Remark 30. We note that it is known that if $c_N(V) < 1$ and $p = 2$ then V is form bounded with relative bound < 1 and $D(A_2) \subset W^{1,2}(R^N)$ (see e.g. [20, Lemma 4.8a, p. 350], [23], [14, Lemma 4] and [40, (2), p. 459]). A more general result is given in Theorem 39 below.

We give now sufficient (smoothness) conditions on the potential to check the main assumption in Theorem 28. (Such conditions are unnecessary if $1 < p \leqslant 2$ and $V \in L^2_{\mathrm{loc}}(R^N)$; see Corollary 40.)

Corollary 31. *Let the general assumptions of Theorem* 8 *be satisfied and let $N \geqslant 2$. We assume that for each $\varphi \in D(\Delta_1) \cap L^\infty(R^N)$*

$$\sup_{|h|\leqslant 1,\, h\neq 0} \left\| \left(\frac{V_h - V}{|h|} \right) \varphi \right\|_{L^1(R^N)} < \infty \tag{19}$$

(*for instance, $V \in W^{1,r}(R^N) + W^{1,s}(R^N) + BV(R^N)$ for some r and s such that* $1 \leqslant r, s \leqslant \infty$). *Then the solution f of* (18) *belongs to $W^{1,p}(R^N)$ for all $1 \leqslant p < \frac{N}{N-2}$. If additionally $V \in L^p_{\mathrm{loc}}(R^N)$ then $\mathcal{C}_c^\infty(R^N)$ is a core for A_p.*

Proof. We note that we already know that $f \in W^{1,1}(R^N)$ since $D(\Delta_1) \subset W^{1,1}(R^N)$ (see [45, p. 65]). Moreover, since $G_\lambda \in L^p(R^N)$ for all p such that $1 \leqslant p < \frac{N}{N-2}$ (see [45, p. 65]) then $D(\Delta_1) \subset L^p(R^N)$ for all p such that $1 \leqslant p < \frac{N}{N-2}$. Moreover (see the proof of Theorem 28), we also know that $f \in L^\infty(R^N)$ because $g \in L^\infty(R^N)$. Thus $f \in L^s(R^N)$ for all $s \in [1, \infty]$.

We observe first that for each vector $h \in R^N$, the translated potential $V_h : x \to V(x+h)$ satisfies also the general assumption (6). It is easy to see by induction on the integer k that for any $\varphi \in L^1(R^N)$

$$\left\| \left[V_h (\lambda - \Delta_1)^{-1} \right]^k \varphi \right\|_{L^1(R^N)} = \left\| \left[V (\lambda - \Delta_1)^{-1} \right]^k \varphi_{-h} \right\|_{L^1(R^N)}.$$

It follows that $\|[V_h(\lambda - \Delta_1)^{-1}]^k\|_{\mathcal{L}(L^1(R^N))} = \|[V(\lambda - \Delta_1)^{-1}]^k\|_{\mathcal{L}(L^1(R^N))}$ and consequently $r_\sigma[V_h(\lambda - \Delta_1)^{-1}] = r_\sigma[V(\lambda - \Delta_1)^{-1}]$ so that the general assumptions of Theorem 8 are satisfied if we replace V by V_h. It follows from (18) that

$$\lambda \left(\frac{f_h - f}{|h|} \right) - \Delta \left(\frac{f_h - f}{|h|} \right) - V_h \left(\frac{f_h - f}{|h|} \right) = \left(\frac{g_h - g}{|h|} \right) + \left(\frac{V_h - V}{|h|} \right) f.$$

We note that $f \in D(\Delta_1) \cap L^\infty(R^N)$ so that, by (19), $\|(\frac{V_h - V}{|h|}) f\|_{L^1(R^N)}$ is uniformly bounded in $|h| \leqslant 1$. Moreover, since $g \in \mathcal{S}(R^N)$ then a simple computation shows that $\|(\frac{g_h - g}{|h|})\|_{L^1(R^N)}$ is also uniformly bounded in $|h| \leqslant 1$. Since V_h is Δ_1-bounded with $r_\sigma[V_h(\lambda - \Delta_1)^{-1}] < 1$ and $\frac{f_h - f}{|h|} \in D(\Delta_1)$ then the property $D(\Delta_1) \subset L^p(R^N)$ for all $1 \leqslant p < \frac{N}{N-2}$ implies the existence of a constant c_p (independent of h with $|h| \leqslant 1$) such that

$$\left\| \frac{f_h - f}{|h|} \right\|_{L^p(R^N)} \leqslant c_p \left[\left\| \left(\frac{g_h - g}{|h|} \right) \right\|_{L^1(R^N)} + \left\| \left(\frac{V_h - V}{|h|} \right) f \right\|_{L^1(R^N)} \right]$$

so that $\sup_{|h|\leqslant 1, h\neq 0} \|\frac{f_h - f}{|h|}\|_{L^p(R^N)} < \infty$. When $p > 1$, this last estimate characterizes the membership of f to $W^{1,p}(R^N)$ (see e.g. [7, Proposition IX.3, p. 153]). Finally, we note that if (for instance) $V \in W^{1,r}(R^N)$ for some $r \in [1, \infty]$ then

$$\sup_{|h|\leqslant 1,\, h\neq 0} \left\| \frac{V_h - V}{|h|} \right\|_{L^r(R^N)} < \infty$$

(see e.g. [7, Proposition IX.3 and Remarque 6, p. 153]) so that

$$\sup_{|h|\leqslant 1,\, h\neq 0}\left\|\left(\frac{V_h - V}{|h|}\right)\varphi\right\|_{L^1(R^N)} \leqslant \sup_{|h|\leqslant 1,\, h\neq 0}\left\|\frac{V_h - V}{|h|}\right\|_{L^r(R^N)}\|\varphi\|_{L^{r'}(R^N)} < \infty,$$

where r' is the conjugate exponent of r and then (19) is satisfied. Similarly, if $V \in BV(R^N)$ then

$$\sup_{|h|\leqslant 1,\, h\neq 0}\left\|\frac{V_h - V}{|h|}\right\|_{L^1(R^N)} < \infty$$

(see e.g. [7, Proposition IX.3 and Remarque 6, p. 153]) and then

$$\sup_{|h|\leqslant 1,\, h\neq 0}\left\|\left(\frac{V_h - V}{|h|}\right)\varphi\right\|_{L^1(R^N)} \leqslant \sup_{|h|\leqslant 1,\, h\neq 0}\left\|\frac{V_h - V}{|h|}\right\|_{L^1(R^N)}\|\varphi\|_{L^{\infty}(R^N)} < \infty$$

and this ends the proof. □

Remark 32. It is easy to drop the condition $p < \frac{N}{N-2}$ by imposing stronger assumptions on the potential; e.g. if for some $r > 1$ and all $\varphi \in D(\Delta_1) \cap L^\infty(R^N)$

$$\sup_{|h|\leqslant 1,\, h\neq 0}\left\|\left(\frac{V_h - V}{|h|}\right)\varphi\right\|_{L^1(R^N)\cap L^r(R^N)} < \infty$$

then $f \in W^{1,p}(R^N)$ for all $p \in [1, r]$.

Under stronger assumptions, we have of course more information on the domain of A_p. Indeed:

Theorem 33. *Let the general assumptions of Theorem* 8 *be satisfied and let $r > 1$. We assume that V is Δ_r-bounded and* (*the spectral radius in $L^r(R^N)$*) $r_\sigma[V(\lambda - \Delta_r)^{-1}] < 1$ *for large* λ. *Then*:

(i) *For all $p \in]1, r]$, $D(A_p) = W^{2,p}(R^N)$ and*

$$A_p f = \Delta f + Vf, \quad f \in D(A_p).$$

(ii) *In particular, if $r \geqslant 2$ then $A_2 : H^2(R^N) \to L^2(R^N)$ is self-adjoint.*

Proof. It follows from Riesz–Thorin interpolation theorem that $V(\lambda - \Delta_p)^{-1}$ is bounded in $L^p(R^N)$ for $p \in [1, r]$. If we choose λ large and m large enough so that $\|[V(\lambda - \Delta_r)^{-1}]^m\|_{\mathcal{L}(L^r(R^N))} < 1$ and $\|[V(\lambda - \Delta_1)^{-1}]^m\|_{\mathcal{L}(L^1(R^N))} < 1$ then, by interpolation, $\|[V(\lambda - \Delta_p)^{-1}]^m\|_{\mathcal{L}(L^p(R^N))} < 1$ (for large λ) for all $p \in [1, r]$. Hence, one sees that for $g \in L^p(R^N)$ the series (17) converges in $L^p(R^N)$. Thus, if we resume the arguments in the proof of Theorem 23 one sees that $(Vf_k) \subset L^p(R^N)$ and converges in $L^p(R^N)$ to Vf and then (Δf_k) converges in $L^p(R^N)$ showing thus that $\Delta f \in L^p(R^N)$, $Vf \in L^p(R^N)$ and $A_p f = \Delta f + Vf$. Finally, for $p \in]1, r]$, the fact that $f \in W^{2,p}(R^N)$ is a standard result in elliptic regularity. □

Remark 34. The fact that if V is Δ_r-bounded with $r_\sigma[V(\lambda-\Delta_r)^{-1}]<1$ for large λ then $\Delta+V$: $W^{2,r}(R^N)\to L^r(R^N)$ is a generator of a (positive) semigroup is a known result relying on different arguments; see [4,23] (see also Remark 44 below). Thus, we find again the known result according to which $C_c^\infty(R^N)$ is a core for A_p if $V\in L^p_{\mathrm{loc}}(R^N)$ and V is (L^p) Δ-bounded with a relative bound $<\frac{1}{2}$ [37]. Note that if $V\in L^r(R^N)$ for some finite $r>\frac{N}{2}$ with $r\geqslant 2$ then Theorem 33(ii) applies and we find again a classical result [16, Theorem 5.1].

5. Connection with form-perturbation theory

In this section we provide connections between our formalism and standard form-perturbation theory.

Theorem 35. *Let $\delta:=\lim_{\lambda\to+\infty}r_\sigma[V(\lambda-\Delta_1)^{-1}]<1$ be satisfied and let $V\in L^2_{\mathrm{loc}}(R^N)$. Then V is form-bounded with respect to $-\Delta$ in $L^2(R^N)$ with relative form-bound $\leqslant\delta$. In particular, if $\lim_{\lambda\to+\infty}r_\sigma[V(\lambda-\Delta_1)^{-1}]=0$ then the relative form-bound of V with respect to $-\Delta$ is equal to zero.*

Proof. Let c be such that $1<c<\frac{1}{\delta}$ and $V_c=cV$. Then

$$\lim_{\lambda\to+\infty}r_\sigma\big[V_c(\lambda-\Delta_1)^{-1}\big]=c\delta<1.$$

According to Theorems 8, 23 and Corollary 24,

$$A_{1c}:\varphi\in D(\Delta_1)\to\Delta\varphi+V_c\varphi\in L^1\big(R^N\big)$$

generates a positive semigroup $\{S_{1c}(t);t\geqslant 0\}$ in $L^1(R^N)$ which interpolates to all L^p spaces providing c_0-semigroups $\{S_{pc}(t);t\geqslant 0\}$ with generateur A_{pc} where A_{2c} is self-adjoint. Moreover,

$$\Xi_{pc}:=\big\{f\in L^p\big(R^N\big)\cap D(\Delta_1);\ \Delta\varphi+V_c\varphi\in L^p\big(R^N\big)\big\}$$

is a core for A_{pc}. Let

$$s_c=\sup\big\{\lambda;\ \lambda\in\sigma(A_{2c})\big\}$$

be the spectral bound of the self-adjoint operator A_{2c}. We note that if $V\in L^2_{\mathrm{loc}}(R^N)$ then $\mathcal{C}_c^\infty(R^N)\subset D(A_{2c})$ and then

$$(A_{2c}\varphi,\varphi)\leqslant s_c\|\varphi\|^2,\quad \varphi\in\mathcal{C}_c^\infty\big(R^N\big),$$

i.e. (taking φ real)

$$-\int|\nabla\varphi|^2\,dx+c\int V\varphi^2\,dx\leqslant s_c\int\varphi^2\,dx,\quad \varphi\in\mathcal{C}_c^\infty\big(R^N\big)$$

so that (putting $\alpha=\frac{1}{c}$)

$$\int V\varphi^2\,dx \leqslant \alpha \int |\nabla\varphi|^2\,dx + \alpha s_{\frac{1}{\alpha}} \|\varphi\|^2, \quad \varphi \in \mathcal{C}_c^\infty(R^N),\ \delta < \alpha < 1.$$

Finally, the density of $\mathcal{C}_c^\infty(R^N)$ in $W^{1,2}(R^N)$ implies

$$\int V\varphi^2\,dx \leqslant \alpha \int |\nabla\varphi|^2\,dx + \alpha s_{\frac{1}{\alpha}} \|\varphi\|^2, \quad \varphi \in W^{1,2}(R^N),\ \delta < \alpha < 1, \tag{20}$$

which ends the proof since α can be chosen as close to δ as we want. □

Corollary 36. *Let V be Δ-weakly compact in $L^1(R^N)$. Then the form-bound of V with respect to $-\Delta$ in $L^2(R^N)$ is equal to zero.*

Proof. The proof of Theorems 4 and 6 show that if V is Δ-weakly compact in $L^1(R^N)$ then $\lim_{\lambda\to+\infty} r_\sigma[V(\lambda-\Delta_1)^{-1}] = 0$. □

According to Theorem 35 the potential V is form-small with respect to $-\Delta$ in $L^2(R^N)$. Then, by the classical KLMN theorem (see e.g. [33, p. 167]) we can define (by means of a closed hermitian form) a self-adjoint operator in $L^2(R^N)$

$$\Delta \dotplus V \quad \text{(form-sum)}.$$

We are going to show that $A_2 = \Delta \dotplus V$. To this end, we need some preliminary results.

Lemma 37. *Let $\varphi \in D(\Delta_1) \cap L^\infty(R^N)$. Then $\varphi \in W^{1,2}(R^N)$.*

Proof. We known (see e.g. the proof of Theorem 26) that $\mathcal{C}_c^\infty(R^N)$ is a core for Δ_1; then there exists $\{\varphi_n\} \subset \mathcal{C}_c^\infty(R^N)$ such that $\varphi_n \to \varphi$ and $\Delta\varphi_n \to \Delta\varphi$ in L^1 norm. A simple inspection of the proof of Theorem 26 shows that if additionally $\varphi \in L^\infty(R^N)$ then the sequence $\{\varphi_n\}$ is uniformly bounded in $L^\infty(R^N)$. On the other hand, the estimate

$$\int |\nabla\varphi_n|^2 = -\int \varphi_n \Delta\varphi_n \leqslant \sup_{k\in N} \|\varphi_k\|_{L^\infty} \|\Delta\varphi_n\|_{L^1}$$

shows that $\{\nabla\varphi_n\}_n$ has a weakly convergent subsequence in $L^2(R^N)$. This shows that $\varphi \in W^{1,2}(R^N)$ since we already know (by interpolation) that $\varphi \in L^p(R^N)$ for all $p \in [1,+\infty]$. □

Lemma 38. *If $V \in L^2_{\mathrm{loc}}(R^N)$ then $\mathcal{C}_c^\infty(R^N)$ is a core for A_2.*

Proof. According to Theorem 28, $\mathcal{C}_c^\infty(R^N)$ is a core for A_2 if the solution $\varphi \in D(A_2)$ to

$$\lambda\varphi - A_2\varphi = g$$

with $g \in \mathcal{S}(R^N)$ (the Schwartz space) has a gradient $\nabla\varphi \in L^2(R^N)$. But for $g \in \mathcal{S}(R^N)$ we have $\varphi \in D(\Delta_1) \cap L^\infty(R^N)$ and then Lemma 37 ends the proof. □

We are ready to show:

Theorem 39. *Let* $\lim_{\lambda\to+\infty} r_\sigma[V(\lambda-\Delta_1)^{-1}]<1$ *and let* $V\in L^2_{\mathrm{loc}}(R^N)$. *Then* $A_2=\Delta\dot{+}V$ (*form-sum*); *in particular* $D(A_2)\subset W^{1,2}(R^N)$ *and*

$$(A_2\varphi,\psi)=-\int\nabla\varphi.\nabla\psi+\int V\varphi\psi,\quad \varphi\in D(A_2),\ \psi\in W^{1,2}\big(R^N\big).$$

Proof. Note that according to Theorem 35, V is form-bounded with respect to $-\Delta$ in $L^2(R^N)$ whence $\int V\varphi\psi$ is well defined for $\varphi,\psi\in W^{1,2}(R^N)$. Let $\varphi\in D(A_2)$ (φ real) and $\{\varphi_n\}\subset\mathcal{C}_c^\infty(R^N)$ converging to φ in the graph norm of A_2, i.e. $\varphi_n\to\varphi$ and $A_2\varphi_n\to A_2\varphi$ in L^2 norm. Let $\lambda>0$. Note first that

$$\lambda\varphi_n-\Delta\varphi_n-V\varphi_n=(\lambda-A_2)\varphi_n$$

implies

$$\lambda\int\varphi_n\phi+\int\nabla\varphi_n.\nabla\phi-\int V\varphi_n\phi=\int\big[(\lambda-A_2)\varphi_n\big]\phi,\quad \phi\in W^{1,2}\big(R^N\big).\tag{21}$$

On the other hand, according to (20),

$$\int V\varphi_n^2\,dx\leqslant\alpha\int|\nabla\varphi_n|^2\,dx+\alpha s_{\frac{1}{\alpha}}\|\varphi_n\|^2,\quad \delta<\alpha<1,$$

so that

$$(\lambda-\alpha s_{\frac{1}{\alpha}})\|\varphi_n\|^2+(1-\alpha)\int|\nabla\varphi_n|^2\,dx\leqslant\big((\lambda-A_2)\varphi_n,\varphi_n\big)$$

and the choice $\lambda>\alpha s_{\frac{1}{\alpha}}$ show that $\{\nabla\varphi_n\}_n$ is bounded in $L^2(R^N)$ and then standard arguments show that $\varphi\in W^{1,2}(R^N)$. Taking a subsequence if necessary, we can pass to the limit in (21) and obtain

$$-\int\nabla\varphi.\nabla\phi+\int V\varphi\phi=(A_2\varphi,\psi),\quad \varphi\in D(A_2),\ \phi\in W^{1,2}\big(R^N\big)$$

which characterizes the form-sum operator $\Delta\dot{+}V$. □

Corollary 40. *Let* $V\in L^2_{\mathrm{loc}}(R^N)$. *Then* $D(A_p)\subset W^{1,p}(R^N)$ *and* $\mathcal{C}_c^\infty(R^N)$ *is a core for* A_p *for all* $p\in[1,2]$.

Proof. We already know this result for $p=1$ (Theorem 26) and for $p=2$ (Lemmas 37 and 38). The bounded operators $(\lambda-A_1)^{-1}$ and $(\lambda-A_2)^{-1}$ (respectively in $L^1(R^N)$ and $L^2(R^N)$) coincide on $L^1(R^N)\cap L^2(R^N)$ and, for all $1\leqslant j\leqslant N$, the bounded operators $\partial_j(\lambda-A_1)^{-1}$ and $\partial_j(\lambda-A_2)^{-1}$ (respectively in $L^1(R^N)$ and $L^2(R^N)$) coincide on $L^1(R^N)\cap L^2(R^N)$ where $\partial_j:=\frac{\partial}{\partial x_j}$. Then Riesz–Thorin interpolation theorem shows that $\partial_j(\lambda-A_1)^{-1}$ interpolates to $L^p(R^N)$ spaces with $p\in[1,2]$ showing thus that $D(A_p)\subset W^{1,p}(R^N)$; finally Theorem 28 ends the proof. □

Remark 41. If $p > 2$ or if $1 < p < 2$ and $V \in L^p_{\text{loc}}(R^N) \setminus L^2_{\text{loc}}(R^N)$, Corollary 40 above does not apply a priori but we can still prove that $\mathcal{C}_c^\infty(R^N)$ is a core for A_p under an additional "smoothness" assumption on V (see Corollary 31).

We end this section with a remark and a conjecture. It is known (see [26, Theorem 3.2]) that if $V \in L^2_{\text{loc}}(R^N)$ is form-bounded with respect to $-\Delta$ in $L^2(R^N)$ with a relative form-bound $\widehat{\delta} < 1$ then for each $\beta \in (\widehat{\delta}, 1)$ there exists $c_1(\beta) \in (1, 2)$ such that the positive semigroup generated by $\Delta \dotplus V$ interpolates on L^p for $p \in [c_1(\beta), c_1'(\beta)]$ ($c_1'(\beta) > 2$ is the conjugate exponent of $c_1(\beta)$) with a suitable estimate depending on the choice of β. Thus, the case $p = 1$ seems to be out of reach of this method even if $\widehat{\delta} = 0$. Suppose now that additionally V is Δ-bounded in $L^1(R^N)$ and that *we can show* that the resolvent $(\lambda - (\Delta \dotplus V))^{-1}$ interpolates on $L^p(R^N)$ with $p \in [1, 2]$ in such a way that it acts in $L^1(R^N)$ as $(\lambda - \Delta_1 - V)^{-1}$ (i.e. $(\lambda - (\Delta \dotplus V))^{-1}$ maps $L^1(R^N)$ into $D(\Delta_1)$). Then, by Lemma 2, we can assert that $\lim_{\lambda \to \infty} r_\sigma[V(\lambda - \Delta_1)^{-1}] < 1$. If this argument works for all potentials $V_c := cV$ where c is such that $c\widehat{\delta} < 1$ (note that $V_c := cV$ is *still* form-small with respect to $-\Delta$ in $L^2(R^N)$) then

$$c \lim_{\lambda \to \infty} r_\sigma\left[V(\lambda - \Delta_1)^{-1}\right] < 1$$

for all c such that $c\widehat{\delta} < 1$ and therefore $\widehat{\delta} \geqslant \delta$. This formal observation (combined to Theorem 35) suggests a plausible conjecture.

Conjecture 42. *Let $V \in L^2_{\text{loc}}(R^N)$ be nonnegative and Δ-bounded in $L^1(R^N)$. Then V is form-small with respect to $-\Delta$ in $L^2(R^N)$ (i.e. $\widehat{\delta} < 1$) if and only if* $\lim_{\lambda \to +\infty} r_\sigma[V(\lambda - \Delta_1)^{-1}] < 1$.

Our conjecture is somewhat "corroborated" by the following result given by E.B. Davies and A.M. Hinz [11] (the authors note that the basic idea of this result goes back to [1,40,42]): Let H_0 be a nonnegative self-adjoint operator in $L^2(R^N)$ such that e^{-tH_0} is an integral operator with a "heat kernel" bound (e.g. $H_0 = -\Delta$); let the quadratic form bound

$$V \leqslant \varepsilon H_0 + \beta(\varepsilon), \quad \forall \varepsilon > 0,$$

hold, i.e. V is form-bounded with respect to H_0 with relative form-bound *zero*, and let $\widehat{\beta}(s) := \inf_{s>0}\{\varepsilon s + \beta(\varepsilon)\}$. If

$$\int_c^{+\infty} \frac{\widehat{\beta}(s)}{s^2}\, ds < \infty$$

($c > 0$ is a constant) then $\lim_{\lambda \to +\infty} \|V(\lambda + H_0)^{-1}\|_{\mathcal{L}(L^1)} = 0$, i.e. V is (L^1) H_0 bounded with relative (operator) bound *zero*.

6. On holomorphy of Schrödinger semigroups

It is known that the Schrödinger semigroups $\{S_p(t); t \geqslant 0\}$ are holomorphic if the relative bound of V is small, see e.g. [21,23,37], [33, p. 253] (see also [10,31] for more information). We show here the L^1-holomorphy in our general setting and extend it to general L^p spaces by standard duality and interpolation arguments.

Theorem 43. *Under the general assumption of Theorem* 8, *the Schrödinger semigroups* $\{S_p(t); t \geqslant 0\}$ *are holomorphic.*

Proof. Consider first the case $p = 1$. In this case $A_1 = \Delta_1 + V$. Since Δ_1 generates a holomorphic semigroup, V is Δ_1-bounded and $\Delta_1 + V$ has a positive resolvent then, by [4, Theorem 1.1], $\Delta_1 + V$ generates a holomorphic semigroup $\{S_1(t); t \geqslant 0\}$. (This argument is not linked to L^1 but rather to the holomorphy of the unperturbed semigroup; in particular, it works in L^p spaces provided that V is Δ_p-bounded and $\Delta_p + V$ has a positive resolvent, i.e. $r_\sigma(V(\lambda - \Delta_p)^{-1}) < 1$ for large λ.) We argue now as in [21]: The holomorphy of $\{S_1(t); t \geqslant 0\}$ is characterized by the existence of $M > 0$ and ω large enough such that

$$\left\|(\lambda - A_1)^{-1}\right\| \leqslant \frac{M}{|\lambda|} \quad (\mathrm{Re}\,\lambda \geqslant \omega).$$

The dual operator in $L^\infty(R^N)$ satisfies the same estimate

$$\left\|\left(\lambda - A_1'\right)^{-1}\right\| \leqslant \frac{M}{|\lambda|} \quad (\mathrm{Re}\,\lambda \geqslant \omega).$$

But we know that $(\lambda - A_1)^{-1}$ and $(\lambda - A_1')^{-1}$ coincide on $L^1(R^N) \cap L^\infty(R^N)$ and then, by Riesz–Thorin interpolation theorem,

$$\left\|(\lambda - A_p)^{-1}\right\| \leqslant \frac{M}{|\lambda|} \quad (\mathrm{Re}\,\lambda \geqslant \omega)$$

which shows that $\{S_p(t); t \geqslant 0\}$ is holomorphic. □

Remark 44. If $V \in L^p(R^N)$ for some $p > \frac{N}{2}$ then, by Sobolev imbeddings ($W^{2,p}$ being the domain of Δ_p), V is Δ_p-bounded and $\Delta_p + V$ has a positive resolvent so that $A_p = \Delta_p + V$ generates a holomorphic semigroup; (see [4] for the details). This provides a different proof of Theorem 33(i) and (in the same time) the holomorphy property; see [23,24] and references therein for more information on Δ_p-bounded potentials.

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